Montgomery's Nine Propositions

In John J. Montgomery's paper "Soaring Flight" that he described to Octave Chanute at the *International Conference on Aerial Navigation* in Chicago, August 1-4, 1893, he lists nine "Propositions" that elucidate the behavior of fluids. They are tutorial in form, beginning simply and becoming more complex. We present them here (in *italics*) and, where possible, provide a clarification of his rather brief expositions (see "*CLARIFICATION*"). Any of his original mathematical derivations and proofs were lost in the Otay Mesa dam break of January of 1916. Considering that he attended St. Ignatius College from 1875 to 1880 (earning both BS and MS degrees) and studied under Joseph Bayma, S.J. (who taught a course that resulted in his 283-page textbook "Infinitesimal Calculus," published in 1889 when he was at Santa Clara College), Montgomery was certainly well-educated enough to develop the necessary mathematics to bolster his Propositions.

Note that Montgomery uses different terms to refer to what is presumably the same object, possibly to emphasize the property that is significant for each Proposition: *body*, *elastic ball*, *fluid particle*, *elastic sphere*, and *particle*.

The varied meanings of a **Proposition**: In mathematics a formal statement of a theorem or problem, typically including the demonstration; sometimes connotes a statement with a simple proof, while the term theorem is usually reserved for the most important results or those with long or difficult proofs; something offered for consideration or acceptance.

PROPOSITION 1ST:

Montgomery begins with some conventional observations on the transmission of an impulse in a fluid based on geometrical considerations.

> *"An impulse generated in a fluid mass is transmitted as a sphere of motion or energy; hence its intensity at any distance is inversely as the square of the distance; corresponding to the increase of surface of a sphere, proportionally to the square of the radii. An impulse transmitted in the fluid, between parallel planes varies in intensity, inversely as the distance; this corresponds to the*

proposition that an arc varies in length proportionally to its distance from the center."

PROPOSITION 2ND:

Montgomery then examines the motion of a body in a fluid as it experiences resistance according to the generally accepted law of drag being proportional to the square of the velocity. He concludes that the strength of the resistance decreases inversely as the square of the time elapsed (as the body slows down) and that this behavior applies both to the case of a body moving through a fluid, or to a stream's effectiveness in acting on a body to impart motion to it.

"A body moving through a fluid encounters resistances in a ratio of its velocity – the proportion is generally supposed to be, as the squares to the velocities.

"An examination of the rate of resistance from the beginning to the cessation of movement reveals, that the resistances are in the inverse ratios of the time. Accepting the statement as correct that resistances are proportional to the squares of the velocity, then the resistances, from the beginning to the end of motion, are inversely proportional to the squares of the times. As it is immaterial whether we consider a body moving through a fluid or a stream acting on a free body immersed in it, this law is equally true. But as the power of a stream to transmit motion to a body, is only the resistance existing between the two, it follows, that the power of a stream to impart motion to an immersed body, there is inversely as the squares of the times from the instant of immersion to the time when the two have the same velocity."

CLARIFICATION:

The drag ("resistance") experienced by a body moving in a fluid at high velocities has the generally accepted form: $F_D = 1/2\, \rho v^2 C_d A$, where ρ is the density of the fluid, v is the speed of the body relative to the fluid, C_d is the body's drag coefficient, and

A is its reference area. For convenience, define $\alpha \equiv 1/2\, \rho C_d A$ so that $F_D = \alpha\, v^2$. Then by Newton's 2nd law

$$m\, dv(t)/dt = -F_D = -\alpha\, v^2(t)\ .$$

Rearranging, we obtain the solution

$$\int_{v(0)}^{v(t)} \frac{dv(t)}{v^2(t)} = -(\alpha / m) \int_0^t dt$$

$$1/v(0) - 1/v(t) = -(\alpha/m) \cdot t$$

or

$$v(t) = v(0) / [\ 1 + (\alpha/m) \cdot v(0) \cdot t\ ]\ .$$

Therefore, as Montgomery states, the "resistance is inversely proportional to the square of the time":

$$F_D = \alpha\, v^2(t) = \alpha \cdot \{\ v(0) / [\ 1 + (\alpha/m) \cdot v(0) \cdot t\ ]\ \}^2 \sim 1/t^2$$

PROPOSITION 3 RD:

Montgomery next considers two cases to illustrate the motion of a body under the influence of forces. In the first case, a body in free space (not within a fluid) is projected across a constant force field. In the second, more relevant, case the body is projected within and across a moving fluid or stream, experiencing both the force of the stream and the resistances of the fluid. In both cases the path of the body is shown to be parabolic (exactly in the first case and approximately in the second case).

> *"If a body moves with a constant velocity, across parallel lines of equal unvarying force, it receives a constant increase of motion or acceleration in the direction of this force. This acceleration will drive it through spaces proportional to the squares of the times. The path thus described by the body, having these two movements, is parabolic, (shown by reference to conic sections).*
>
> *"When a body is projected across a stream, it continually cuts parallel lines of force and should describe a parabolic curve; however, owing to the fluid resistances it loses motion, inversely as the squares of the times of motion. This destruction of its velocity would prevent its traveling in a parabolic path, but for the further fact, that the intensity of the parallel lines of force (of the stream)*

which carry it along, varies in the same proportion. Hence the relations of the controlling forces remain the same as those of a constant velocity and regular increase of motion.

"This law applies equally to a single impulse or to a number, following one another in rapid succession, as those of a fluid stream or jet.

"However, an important difference in results follows from the application of a force in these two ways."

CLARIFICATION:

Case 1 – Body projected across parallel lines of equal unvarying force:

Define: y = position <u>across</u> lines of force, x = position <u>along</u> lines of force.

The body experiences no forces in the y-direction so that its velocity, v_y, in that direction is constant and its position varies linearly with time

$$y(t) = v_y \bullet t\ .$$

In the x-direction, however, the body is accelerated by the constant force F

$$m\ dv_x(t)/dt = F$$

with the solution

$$\int_0^{v_x(t)} dv_x(t) = (F/m)\int_0^t dt$$

$$v_x(t) = (F/m)\bullet t$$

and its position can be determined from the definition of velocity

$$v_x(t) = dx(t)/dt$$

$$\int_0^{x(t)} dx(t) = (F/m)\int_0^t t \bullet dt$$

and is seen to vary quadratically in time

$$x(t) = (1/2)\bullet(F/m)\bullet t^2\ .$$

Therefore the relationship between position x(t) and y(t) is

$$x(y) = (1/2)\bullet[\ F/(m\ v_y^2)\ ]\bullet y^2$$

which is parabolic, as stated by Montgomery. The parabolic parameter a (the distance from the vertex to either the directrix or focus) in the standard equation for a parabola, $4ax = y^2$, is $a = (1/2\ m\ v_y^2) / F$

Case 2 – Body projected across a stream

Define: y = position <u>across</u> stream, x = position <u>along</u> stream.

Define: v_s = constant stream velocity, positive in the x-direction.

The moving body will experience drag in both the x- and y-directions, acting to slow it down. In addition, the body will feel the force of the moving stream (as a "negative" drag) to speed it up in the x-direction.

Hence, in the y-direction (across the stream), the body will experience deceleration due to the drag force, F_D ,

$$m\, dv_y(t)/dt = -F_D = -\alpha\, v_y^2(t)$$

which has the solution (see prop. 2)

$$v_y(t) = v_y(0) / [\ 1 + (\alpha/m)\bullet v_y(0)\bullet t\ ] .$$

The y-position can then be found from the definition of velocity

$$v_y(t) = dy(t)/dt$$

so that

$$\int_0^{y(t)} dy(t) = v_y(0) \int_0^t \frac{dt}{[1 + (\alpha / m) v_y(0) \bullet t]}$$

$$y(t) = (m/\alpha)\bullet \ln[\ 1 + (\alpha /m)\ v_y(0)\bullet t] .$$

In the x-direction (along the stream), there are two forces acting on the body: that due to the flowing stream, F_s (positive), and that due to drag, F_D (negative),

$$m\, dv_x(t)/dt = F_s - F_D = \alpha\, v_s^2 - \alpha\, v_x^2(t).$$

which has the solution

$$\int_0^{v_x(t)} \frac{dv_x(t)}{v_s^2 - v_x^2(t)} = -(\alpha / m) \int_0^t dt$$

so that

$$v_x(t) = v_s \bullet \{\exp[(2\alpha v_s/m)\bullet t] - 1\} \ / \ \{\exp[(2\alpha v_s/m)\bullet t] + 1]\} \ .$$

The x-position can be found from the definition of velocity

$$v_x(t) = dx(t)/dt$$

so that

$$\int_0^{x(t)} dx(t) = v_s \int_0^t \frac{\exp[(2\alpha v_s / m) \bullet t] -1}{\exp[(2\alpha v_s / m) \bullet t] +1} \, dt$$

which has the solution

$$x(t) = (m/2\alpha)\bullet \ln\{ [\exp[-(2\alpha v_s/m)\bullet t] \ + 1]\bullet [\exp[(2\alpha v_s/m)\bullet t] + 1]/4\} \ .$$

The relationship between x(t) and y(t) can be shown to be quadratic for small values of t by noting that the functions $\ln(1\pm u) \approx \pm u$ and $\exp(\pm u) \approx 1 \pm u$ for $u<<1$. Therefore, for small values of t, y(t) and x(t) can be approximated by

$$y(t) \approx v_y(0)\bullet t$$

$$x(t) \approx -(\alpha/2m)\bullet \ v_s^2 \bullet t^2$$

so that the relation between x(t) and y(t) is

$$x(y) \approx (\alpha/2m)\bullet \ [v_s/v_y(0)]^2 \bullet y^2 \ \sim \ y^2 \text{ (for small values of t)}$$

which is parabolic, as stated by Montgomery.

We have found that, for larger values of t, the relationship between x(t) and y(t) is approximately quadratic (see following example).

Example of a ball projected into a stream of water:

As an example, consider the case of a 1-inch (2.54-cm) smooth glass ball projected (submerged) with a velocity of 1 mph (45 cm/s) across a stream that is flowing at 2 mph. (We ignore the effects of gravity.) This glass ball has the properties:

m (mass) = 21.5 gm (for a density of 2.5 gm/cm^3, typical for glass)

A (reference area) = $\pi r^2 = 5.067$ cm^2 (where r is the ball's radius)

C_d (drag coefficient) = 0.10 (for a smooth sphere).

Since the density of water is ρ = 1.0 gm/cm^3 the parameter α in the above equations becomes

$$\alpha = 1/2\ \rho C_d A = 0.245 \text{ gm/cm}$$

The figures below show the exact solution for this case. The first figure shows the velocity histories of the ball in the x- and y-directions. We see that the ball slows down as it crosses the stream (from 45 to 20 cm/s) but is sped up by the stream, approaching the velocity of the stream (90 cm/s). The second figure shows the position of the ball as a function of time as an x-y plot (how the ball's motion would appear from above). We see that the x- and y-positions appear to be related quadratically and a power-law fit to the solution yields the following relationship, with an exponent very close to 2 (within 5%):

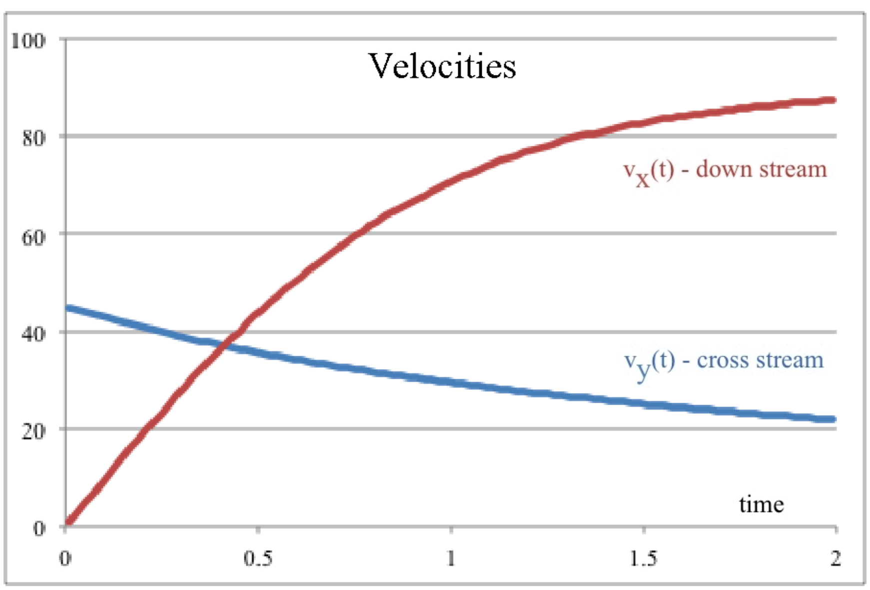
$$x(y) = 0.0219 \cdot y^{2.097} .$$

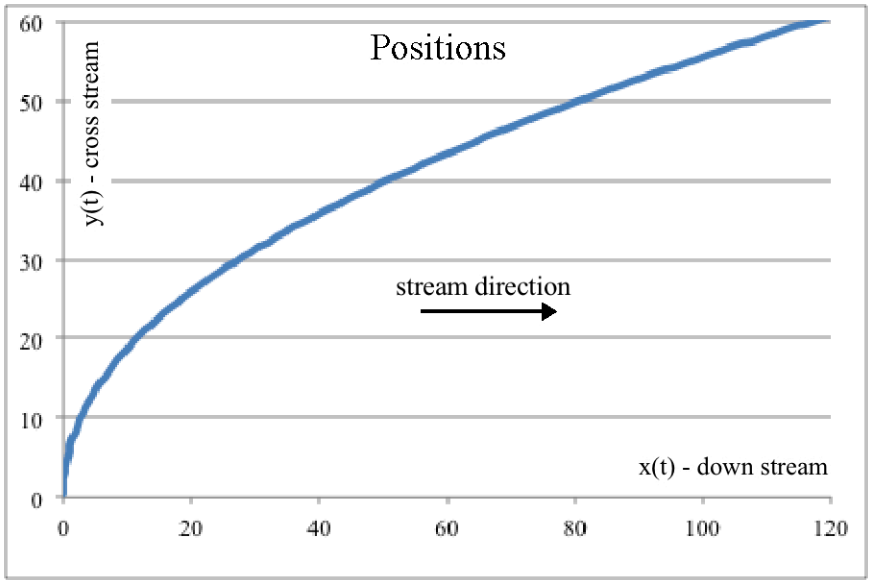
(The fitted curve is not shown since it accurately overlays the exact solution.)

Referring to the analytic solution derived above for small t,

$$x(y) \approx (\alpha/2m) \cdot [v_s/v_y(0)]^2 \cdot y^2 = 0.0228 \cdot y^2 ,$$

we see that the constant term is about the same as the power-law fit of the full solution. Montgomery's conjecture is very accurate: "When a body is projected across a stream, it continually cuts parallel lines of force and should describe a parabolic curve… "





PROPOSITION 4 TH:

Montgomery makes the following observations concerning the two cases presented in Prop. 3: Both the energies of a body projected into a fluid and the flowing stream rebounding from the body are converted into whirls within the fluid, forming parabolic curves. He explores this more fully in Prop. 5.

> *"In the first manner of projection, a set of whirls is formed by which the energy of the moving body is absorbed; in the second, the energy is transformed into a counter movement, which in reality, is the rebounding of the jet from the forward resistances. Both the direct and counter movement of this jet come under the previous law, forming opposite branches of parabolic curves, as will appear from the following."*

PROPOSITION 5 TH:

Montgomery next considers the case of multiple collisions of an elastic ball (viz., a fluid particle) in a uniform force field (viz., gravity or fluid stream) and concludes it will travel on parabolic paths after each rebound from either a surface or other bodies (fluid particles) according to Props. 3 and 4.

"An elastic ball 'a' projected horizontally and being pulled down by gravity, will trace the curve 'abc' (Fig. 5).

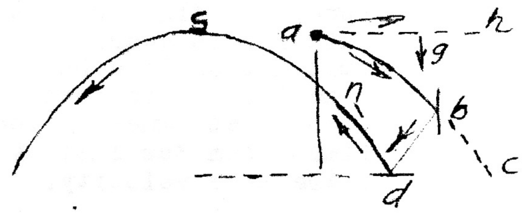


Fig. 5.

"At 'b' it meets a reflecting surface and rebounds in direction 'bd'. At 'd' it meets another plane which, by reflection sends it in the direction 'dn'; but being controlled by gravity, it traces the curve 'ds', similar to 'abc'. Reaching the point 'S' it again descends, forming a curve the counterpart of 'abc'.

"Applying this to the movements of a fluid particle, according to the conditions stated in prop. (3 & 4), we find that a fluid mass projected across a current will form parabolic lines in the first impulse and also in bounding from the opposing particles. As these movements take place in the stream, the curves which are formed from the direct and reflected impulses will be somewhat as represented in Fig. (6)."

Fig. 6

PROPOSITION 6 TH:

Montgomery, having explored the behavior of single elastic spheres, next explores, in this and the following Proposition, the behavior of multiple conjoined elastic spheres reflecting from a solid surface. With these examples he illustrates that conjoined elastic spheres reflect quite differently from a surface than single elastic spheres.

This Proposition considers the case of three adjacent elastic spheres in a specific triangular configuration, moving as a unit. Montgomery shows that, when reflected from a surface, they end up separating and moving in different directions, with two of them moving parallel to the surface.

"A fluid meeting a solid is reflected essentially in three directions, as will be apparent from the examination of Fig. (7).

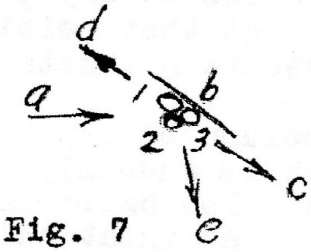


Fig. 7

"Let three elastic spheres '1; 2; 3', have the motion 'a', impinge against the plane 'b'; '1' and '3' will be reflected from the surface according to the established laws, but coming in contact with '2' are reflected from its surface in the direction 'c' and 'd' while '2'is thrown in the direction 'e'."

<u>PROPOSITION 7 TH</u>:

This Proposition considers the case of two separated but connected elastic spheres. Montgomery shows that, when one of them is reflected from a surface, the two spheres end up rotating in opposite directions about a common center (remaining connected together).

"Any two points taken in the circumference of a rotating circle, at any instant, are found to be moving in different directions; conversely, if either of two points, connected by any means which tends to keep them the same distance apart, is forced to move in a different direction to that of the other, they will move in circular paths around a common center, as illustrated in the following case (Fig. 8).

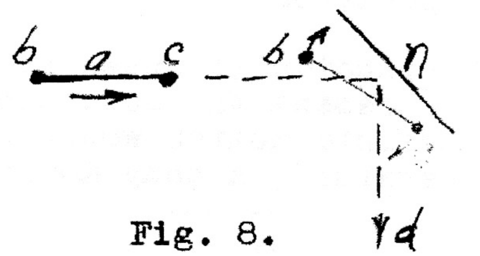


Fig. 8.

"Let two spheres be attached to a rod 'a' having the indicated motion; when 'C' which is elastic comes in contact with the plane 'p', it will be reflected in direction 'd', and reacting on the sphere 'b', will change its direction of motion, while it, in turn is deflected from its path 'd'. And the two thus moving in different directions connected by the rod 'a', describes circular paths around a common center."

<u>PROPOSITION 8 TH</u>:

We have redrawn the original figure for clarity.

"Let 'abdg' be a column of liquid, resting on the base 'ab'.

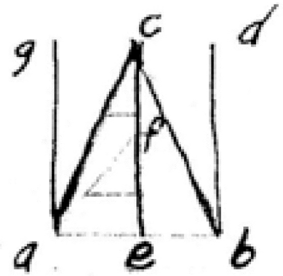
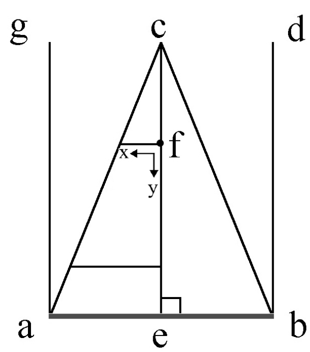


Fig. 9.



"Draw the lines 'ce', 'ac', and 'bc' ('ce' being perpendicular to 'ab' at its center).

"As the pressure in the column increases with its depth, a perpendicular drawn at any point, from 'ce' to 'ac' will represent the pressure at that point. The increase of pressure from 'c' to 'e' represents a constant and regular increase of force.

"Let this column be replaced by one of elastic material having motion, which

is suddenly arrested by the base 'ab', then the pressure will also be proportional to the depth; for the pressure is expressed in terms of the velocity and mass; but the velocity is the same for every part of the column, hence the pressure at any point 'ce' is proportional to the mass pressing against it.

"If this column is one of fluid, having indefinite length and breadth, moving between two parallel planes, resting on the opposite edges of the plane 'ab' the same proportion of pressures exists. In this case the line 'ce' is an indefinite quantity; however, we may assume any point 'f' as the origin of pressures, and calculate the increase of pressure from this point to the base 'ab', just as if no superincumbent pressure existed.

"This moving column of fluid represents a constant force having a constant velocity in the direction 'fe'. If then this force commence to develop work at any point, 'f', the work performed between this point and the plane, will be proportional to the time of operation.

"And a perpendicular drawn from 'fe' to 'fa', at any point, will therefore represent the accumulated work at that point, and which transformed into motion would generate velocities proportional to the time. Therefore, a body moving from 'f' to 'e' with a constant velocity and urged from the line 'fe' by the increasing force, will pass through distances perpendicular to 'fe', proportional to the squares of the times.

"Representing the times, by distances along the line 'fe' (which we shall call y) and the spaces perpendicular to this, through which the body moves, by 'x'; at two consecutive seconds we have the spaces and times represented by the equation $x:x' = y^2:y'^2$.

"This as already stated, is the equation of the parabolic curve. Transforming this into the equation $y^2 = 2px$, *we may determine the curve, when any values of x and y are given. And if a surface be constructed in accordance with the curve corresponding to any point on the line 'ce', and placed in its appropriate position, the fluid particles urged by the advancing column will move over it exerting their full energy.*

"But it will be observed that the particles moving toward 'a' are pressed upon by those moving towards 'b'. If the latter pressure be removed, its reactive influence must be supplied by another force. This may be found by inclining the curve to the advancing column, and decomposing this force into two elements, one parallel with the ordinate (y) and the other (representing the reaction from 'b')) perpendicular to it.

"Other considerations will appear in Prop. (9)."

PROPOSITION 9 TH:

Montgomery further develops his analysis of the behavior of a fluid flowing about an airfoil to show that a parabolic-shaped airfoil is the appropriate surface. We have found his arguments to be quite involved (convoluted?) and difficult to fully comprehend, though they seem to make some sense. Montgomery was certainly able to convince himself. Maybe an interested reader will be able to clarify or condense his explanation.

We have redrawn the original figure for clarity and inserted Fig. 14 from Exp. 2 here since Montgomery refers to it. We have added the point "A" to our redrawn figure based on our interpretation of the context of Montgomery's exposition. It did not appear in the original figure (either Montgomery's handwritten document or Regina's transcription) and is referred to in Montgomery's handwritten document as "a" (not "A"), but appears correctly as "A" in Regina's transcription (corrected in a conversation with her husband?).

"In the parabolic curve the following relations appear. If from the focus 'c' with a radius, equal to the focal length of the parabola, 'ea'; we describe a circle we find that a line drawn from the center of the circle to any point on the curve will be cut at the circumference, so that the portion outside the circle equals the perpendicular distance from 'ed' to the same point. (This readily appears from inspection of Fig. (10) and the construction of the parabolic curve.)

"The line 'ed' is tangent to both the circle and the curve at its vertex 'e'.

"The tangent of point 'a' bisects this at point 'b'.

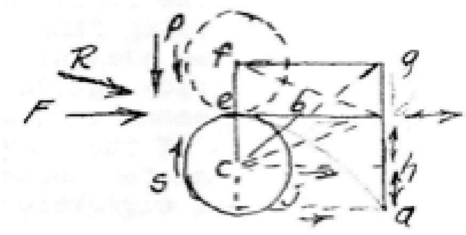
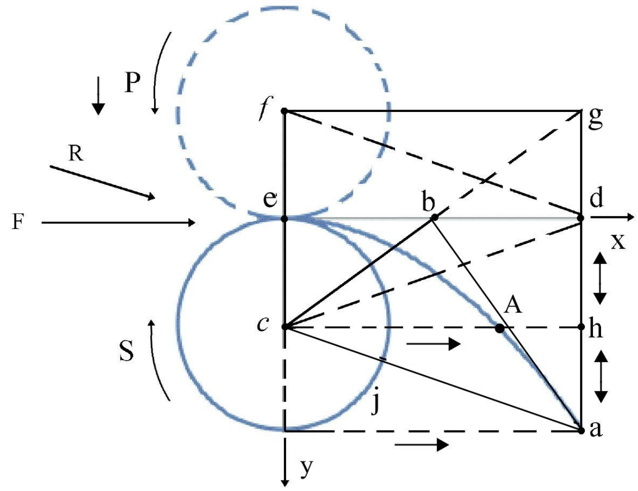


Fig.10.



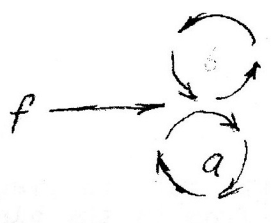


Fig. 14

"Suppose while the circle rotates, in the direction of the arrow, a particle escapes from it at the point 'e', it will travel along the line 'ed', without restraint, and therefore perform no work. But if it is compelled to move with the rotating circle, it is entirely restrained from giving out any of its energy.

"If however it travels along a line equally distant from these two extremes, it will perform its maximum work.

"From this figure, we find that his line is the curve 'ea'.

"This development of work presupposes that the particle is urged from the line 'ed' and the circumference by equal forces. That such is the case, may appear from the following. If we refer to the analysis of fluid reactions in Fig. (14), we find that when an impulse is imparted to a fluid particle, it is opposed by the inertia of those in front of it, and thus are produced pressure and motion perpendicular to the line of impulse, giving rise to two opposite rotations.

"Referring this to Fig. (10), let 'F' represent the impulse, and 'f' and 'c' be

the centers of these rotations. Since these rotations are formed and held in position by the external fluid resistances, they exert a constant and equal pressure in all directions. The line 'ed' is the direction of the force 'F' and is perpendicular to 'cf', the line joining the centers of the two rotations; it then, is the line mutual reactions from both centers.

"Therefore, it may be considered a fixed line upon which either rotation exerts its pressure. Then the following analysis appears.

"The force 'cd' is composed of 'ch' and 'hd'; but the point 'd' being fixed, produces the reaction 'dh'; the force 'ca' has the elements 'ch' and 'ah'. The sum of the reaction 'dh' and the direct action 'ah' is 'ad'. This analysis is general for the pressures on all points along the line 'ed'. Then, for all points of the curve below the line 'ch' the element 'ad' is positive and for those above, it is negative; hence the line 'ad' is the algebraic sum of the elements 'dh' and 'ha'.

"Since the forces 'cd' and 'ca' develop work by pressing on the forward resistances, the point 'A' will be found on the line 'da', whose distance from 'c' is one half the sum of the two elements 'ch'.

"As the point 'A' (according to this analysis) is equally distant from the circumference of the circle and the line 'ed', we infer that both exert the same pressure upon it.

"The converse of this proposition is true, and we may place a plane 'ea' so as to receive the impulses of a current, which in acting upon it, will develop the pressures just discussed, and a tendency to rotation around the focus 'c'.

"From the figure is appears that when the force 'F' produces the two rotations around 'c' and 'f', that the element 'S' of rotation 'c' is kept in balance by the opposing element 'P' of rotation 'f'. Then if this system is built up by the action of a current on a curved surface 'ea', the deficiency of the element 'P' must be supplied by an element of a current having the direction 'R'.

"As this system is developed by the movement of a current along the line 'ed', equal work is produced along equal portions of this line; for the pressure and velocity of the current are constant. And as the point 'b' is in the center of 'ed',

equal work is performed on either side of it.

"This point then becomes the center of movements, the lines 'eb' and 'ed', being the lever arms of a series of perpendicular forces. Then also, the lines 'eb' and 'ab' may be considered lever arms passing through the same point; but these lines are tangents to the extremities of the curve. Hence the pressures on the portions of the curve subtended by these tangents, are inversely as the lengths of their tangents."

<u>FLUID IMPULSES</u>:

"When a force 'f' (Fig. 11) impulsive or continuous imparts motion to a fluid particle, it meets a resistance 'b' due to the inertia of the forward particles.

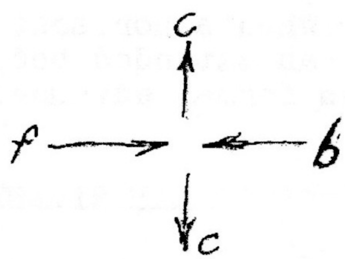


Fig. 11

"The pressure of the opposing forces, causes a series of other forces radiating around the point of contact and perpendicular to the line of motion ('f').

"This forms the basis of all fluid action, finding expression in innumerable phenomena, and hence cannot be too thoroughly studied.

"Each of the resulting radial forces gives rise to a similar set of movements, and these again to others, till a pressure is produced in the surrounding fluid.

"This multitude of forces find a resultant direction of their combined elements and the fluid particles take up a corresponding movement.

"If we examine a section of these forces, the resultants appear as in Fig. (12)

"These resultants give rise to two opposite rotations 'a' and 'b'.

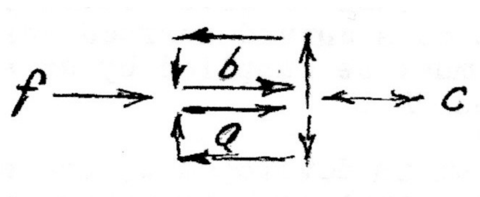


Fig. 12.

"These rotations if radiating around the lines of impulses, in a fluid mass, give rise to those beautiful rings seen when a quick puff comes from an orifice containing smoke. In its incipient state this phenomenon appears on the under rounded surface of tobacco smoke, when sinking in quiet air.

"In liquid surfaces this law of fluid impulse and reaction produces different phenomena according to the nature and direction of movement. If a pebble be dropped on the surface of water, it will cause concentric rings or waves; whose elements or force, are radiations from the point of contact on the surface of the water.

"These different radial elements produce a set of reactions whose analysis is different, according as they are viewed parallel with or perpendicular to the surface of the water.

"When a horizontal impulse is given to the surface of water, by an extended body, a series of concentric semi-curves; of various forms, advances along the surface."